RATIONAL MISIURESWICZ MAPS FOR WHICH THE JULIA SET IS NOT THE WHOLE SPHERE

MAGNUS ASPENBERG

Abstract. We show that Misiurewicz maps for which the Julia set is not the whole sphere are Lebesgue density points of hyperbolic maps.

1. Introduction

In [12] by Rivera-Letelier, it is shown that Misiurewicz maps for unicritical polynomials of the form $f_c(z) = z^d + c$, $c \in \mathbb{C}$, are Lebesgue density points of hyperbolic maps. This paper extends this result to all Misiurewicz maps in the space of rational functions of a given degree $d \geq 2$, if the Julia set is not the whole sphere. i.e. every Misiurewicz map for which $J(f) \neq \hat{\mathbb{C}}$ is a Lebesgue density point of hyperbolic maps. The statement is false if the Julia set is the whole sphere (see e.g. [3]), because in this case the Misiurewicz maps are Lebesgue density points of Collet-Eckmann maps (CE). In addition, these CE-maps have their Julia set equal to the whole sphere (see also [11]).

This paper complements [1], where Misiurewicz maps for which $J(f) = \hat{\mathbb{C}}$ are studied. In particular, it is shown in that paper that every such Misiurewicz map apart from flexible Lattés maps can be approximated by a hyperbolic map. We get the following measure theoretic characterisation: Let f be a rational Misiurewicz map. Then if f is not a flexible Lattés map, there is a hyperbolic map arbitrarily close to f. Moreover,

- if $J(f) = \hat{\mathbb{C}}$, then f is a Lebesgue density point of CE-maps,
- if $J(f) \neq \hat{\mathbb{C}}$, then f is a Lebesgue density point of hyperbolic maps.

The notion of Misiurewicz maps goes back to the famous paper [9] by M. Misiurewicz. In that paper, real maps of an interval are considered and in the complex case there are some variations of the definition of Misiurewicz maps (see e.g. [6], [14]). We proceed with the following definition. First, let $J(f)$ be the Julia set of the function f and $F(f)$ its Fatou set. The set of critical points is denoted by $Crit(f)$ and the omega limit set of x is denoted by $\omega(x)$.

Definition 1.1. A rational non-hyperbolic map f is a *Misiurewicz map* if f has no parabolic periodic points and for every $c \in Crit(f)$ we have $\omega(c) \cap Crit(f) = \emptyset$.

Theorem A. *If f is a rational Misiurewicz map of degree $d \geq 2$, for which $J(f) \neq \hat{\mathbb{C}}$, then f is a Lebesgue density point of hyperbolic maps in the space of rational maps of degree d.*

The space of rational maps of degree d is a complex manifold of dimension $2d+1$. To prove Theorem A we will consider 1-dimensional balls around the starting map f. If $B(0, r)$ is a 1-dimensional ball in the parameter space of rational maps of degree

The author gratefully acknowledges funding from the Research Training Network CODY of the European Commission.

$d \geq 2$, then we can associate a direction vector $v \in \mathbb{P}(\mathbb{C}^{2d})$ to $B(0,r)$, such that the plane in which $B(0,r)$ lies can be parameterized by $\{tv : t \in \mathbb{C}\}$. In this case we say that $B(0,r)$ has direction v.

Theorem A above follows directly from the following.

Theorem B. *Let $r > 0$ and f_a, $a \in B(0,r)$ be a 1-dimensional family of rational functions of degree $d \geq 2$ and suppose that $f = f_0$ is Misiurewicz map for which $J(f) \neq \hat{\mathbb{C}}$. Then for almost all directions v of $B(0,r)$, f is a Lebesgue density point of hyperbolic maps in the ball $B(0,r)$.*

We also note that combining [12] with Theorem A, every Collet-Eckmann map for which the Julia set is not the whole sphere can be approximated by a hyperbolic map. In particular, this holds for all polynomial Collet-Eckmann maps. In view of [12] and [3] it seems natural that almost every Collet-Eckmann map has its Julia set equal to the whole sphere.

Acknowledgements. I am thankful to the referee for many useful remarks. This paper was written at Mathematisches Seminar at Christian-Albrechts Universität zu Kiel. The author gratefully acknowledges the hospitality of the department.

2. Preliminary lemmas

We will use the following lemmas by R. Mañé.

Theorem 2.1 (Mañé's Theorem I). *Let $f : \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$ be a rational map and $\Lambda \subset J(f)$ a compact invariant set not containing critical points or parabolic points. Then either Λ is a hyperbolic set or $\Lambda \cap \omega(c) \neq \emptyset$ for some recurrent critical point c of f.*

Theorem 2.2 (Mañé's Theorem II). *If $x \in J(f)$ is not a parabolic periodic point and does not intersect $\omega(c)$ for some recurrent critical point c, then for every $\varepsilon > 0$, there is a neighborhood U of x such that*

- *For all $n \geq 0$, every connected component of $f^{-n}(U)$ has diameter $\leq \varepsilon$.*
- *There exists $N > 0$ such that for all $n \geq 0$ and every connected component V of $f^{-n}(U)$, the degree of $f^n|_V$ is $\leq N$.*
- *For all $\varepsilon_1 > 0$ there exists $n_0 > 0$, such that every connected component of $f^{-n}(U)$, with $n \geq n_0$, has diameter $\leq \varepsilon_1$.*

An alternative proof of Mañé's Theorem can also be found by L. Tan and M. Shishikura in [13]. Let us also note that a corollary of Mañé's Theorem II is that a Misiurewicz map cannot have any Siegel disks, Herman rings or Cremer points (see [7] or [13]).

For $k \geq 0$, define

$$P^k(f) = \overline{\bigcup_{n>k, c \in Crit(f) \cap J(f))} f^n(c)}.$$

Given a Misiurewicz map f, there is some $k \geq 0$ such that $P^k(f)$ is a compact, forward invariant subset of the Julia set which contains no critical points.

By Mañé's Theorem I, the set $\Lambda = P^k(f)$ is hyperbolic. It is then well-known that there is a holomorphic motion h on Λ:

$$h : \Lambda \times B(0,r) \to \mathbb{C}.$$

For each fixed $a \in B(0,r)$ the map $h = h(z,a) = h_a$ is an injection from Λ to $h_a(\Lambda) = \Lambda_a$ and for fixed $z \in \Lambda$ the map $h = h(z,a)$ is holomorphic in a.

Each critical point $c_j \in J(f)$ moves holomorphically, if it is non-degenerate (i.e. c_j is simple), by the Implicit Function Theorem. If it is degenerate, we have to use a new parameterisation to be able to view each critical point as an analytic function of the parameters. If the parameter space is 1-dimensional one can use the Puiseaux expansion (see e.g. [4] Theorem 1 p. 386). By reparameterising using a simple base change of the form $a \to a^q$ for some integer $q \geq 1$, the critical points then move holomorphically. In the multi-dimensional case, i.e. if we consider the whole $2d-2$-dimensional ball $\mathbb{B}(0,r)$ in the parameter space, a corresponding result is outlined in [1]. Here we restrict ourselves to just state the result (it is a complex analytic version of Lemma 9.4 in [10]). There is a proper, holomorphic map $\psi : U \to V$, where U and V are open sets in $\mathbb{C}^{2d-2}$ containing the origin, such that $f'(z,a)$ can be written as

$$f'(z,\psi(a)) = E(z-c_1(a)) \cdot \ldots \cdot (z-c_{2d-2}(a)),$$

where each $c_j(a)$ is a holomorphic function on U and E is holomorphic and non-vanishing. We therefore assume that all critical points c_j on the Julia set moves holomorphically.

We know that for some $k \geq 0$ we have $v_j := f^{k+1}(c_j) \in \Lambda$ for all $c_j \in Crit(f) \cap J(f)$. Thus we can define the parameter functions

$$x_j(a) = v_j(a) - h_a(v_j(0)).$$

Let $\mathbb{B}(0,r)$ be a full dimensional ball in the parameter space of rational maps around $f = f_0$. Since we already know that Misiurewicz maps cannot carry an invariant line field on its Julia set, (see [2]), not all the functions x_j can be identically equal to zero in $\mathbb{B}(0,r)$.

Lemma 2.3. *If f is a Misiurewicz map then at least one x_j is not identically equal to zero in $\mathbb{B}(0,r)$.*

In fact, it follows a posteriori, that every such x_j is not identically zero. However, let us now assume that I is the set of indices j such that x_j is not identically zero in $\mathbb{B}(0,r)$. We know that $I \neq \emptyset$. In the end, we prove that in fact $I = \{1,\ldots,2d-2\}$.

Hence the sets $\{a : x_j(a) = 0\}$, $j \in I$, are all analytic sets of codimension 1. Hence for almost all directions v the funtions x_j, $j \in I$ are not identically equal to zero in the corresponding disk $B(0,r)$. From now on, fix such a disk $B(0,r)$ for some $r > 0$.

Definition 2.4. Given $0 < k < 1$, a disk $D_0 = B(a_0, r_0) \subset B(0,r)$ is a *k-Whitney disk* if $|a_0|/r_0 = k$.

A Whitney disk is a k-Whitney disk for some $0 < k < 1$.

We will now use a distortion lemma from [2], Lemma 3.5. In this lemma we put $\xi_n = \xi_{n,j}$ and

$$\xi_{n,j}(a) = f_a^n(c_j(a)),$$

where $a \in B(0,r)$. Moreover, choose some $\delta' > 0$, such that $\mathcal{N}$ is a fixed $10\delta'$-neighbourhood of Λ such that $\Lambda_a \subset \mathcal{N}$ for all $a \in B(0,r)$ and $\mathrm{dist}(\Lambda_a, \partial\mathcal{N}) \geq \delta'$. This $\delta' > 0$ shall be fixed throughout the paper and depends only on f.

Lemma 2.5. *Let $\varepsilon > 0$. If $r > 0$ is sufficiently small, there exists a number $0 < k < 1$ only depending on the function x_j, and a number $S = S(\delta')$, such that the following holds for any k-Whitney disk $D_0 = B(a_0, r_0) \subset B(0,r)$: There is an $n > 0$*

such that the set $\xi_n(D_0) \subset \mathcal{N}$ *and has diameter at least* S*. Moreover, we have low argument distortion, i.e.*

$$\left| \frac{\xi_k'(a)}{\xi_k'(b)} - 1 \right| \leq \varepsilon, \tag{1}$$

for all $a, b \in D_0$ *and all* $k \leq n$*.*

Hence, if ε is small, we have good geometry control of the shape of $\xi_n(D_0)$ up to the large scale $S > 0$, i.e. it is almost round. We will use the fact that this holds for every x_j, $j \in I$.

3. Conclusion and proof of Theorem B

We recall the following folklore lemma. For proofs see e.g. [8] (see also [5] for the case of polynomials).

Lemma 3.1. *Let* f *be a Misiurewicz map for which* $J(f) \neq \hat{\mathbb{C}}$*. Then the Lebesgue measure of* $J(f)$ *is zero.*

For each critical point $c_j = c_j(0) \in J(f)$, $j \in I$ put $D_j = \xi_{n_j,j}(D_0)$, where n_j is the number n in Lemma 2.5. Hence for every j, we have that the diameter of D_j is at least S and we have good control of the geometry, if $\varepsilon > 0$ is small in Lemma 2.5.

Next we prove the following lemma.

Lemma 3.2. *For each compact subset* $K \subset F(f)$ *there is a perturbation* $r = r(K)$ *such that* $K \subset F(f_a)$ *for all* $a \in B(0, r)$*.*

Proof. It follows from [13] and [7] that the only Fatou components for Misiurewicz maps are those corresponding to attracting cycles. Recall that $f = f_0$.

Given $K \subset F(f_0)$, there is some integer n and some small disk $B_j \subset F(f_0)$ around each attracting orbit such that $K \subset f_0^{-n}(D)$, where $D = \cup_j B_j$. Choose D such that $f_0(D) \subset D$. Since $f_a(D) \subset D$ for small perturbations $a \in B(0, r)$, we have $f_a^n(D) \subset D$ for all $n \geq 0$. Hence the family $\{f_a^n\}_{n=0}^{\infty}$ is normal on D and consequently $D \subset F(f_a)$ for any such parameter $a \in B(0, r)$. Moreover, $f_a^{-n}(D)$ moves continuously with the parameter, and therefore there is some $r > 0$ such that also $K \subset f_a^{-n}(D)$ for all $a \in B(0, r)$. The lemma is proved. ☐

Let $\delta > 0$. Define

$$E_\delta = \{z \in F(f_0) : dist(z, J(f_0)) \geq \delta\}.$$

Now, there is some $\delta_0 > 0$ (depending only on $f = f_0$) such that for every $0 < \delta < \delta_0$ there exist an $r = r(\delta) > 0$ such that $E_\delta \subset F(f_a)$ or every $a \in B(0, r)$, by Lemma 3.2.

Clearly, $r(\delta) \to 0$ as $\delta \to 0$. Since the Lebesgue measure of $J(f_0)$ is zero, for every $\varepsilon_1 > 0$ there is some $\delta > 0$ such that the Lebesgue measure of the set $\{z : dist(z, J(f_0)) \leq \delta\}$ is less than ε_1. Hence we conclude that there exists some $\delta > 0$ such that for every disk D of diameter at least $S/2$ ($S > 0$ is the large scale from Lemma 2.5) we have

$$\frac{\mu(D \cap E_\delta)}{\mu(D)} \geq 1 - \varepsilon_1.$$

For this $\delta > 0$, there is some $r = r(\delta) > 0$ such that also $E_\delta \subset F(f_a)$, for all $a \in B(0, r)$. Since every D_j contains a disk of diameter $S/2$ (because of bounded distortion), we therefore get

$$\frac{\mu(D_j \cap E_\delta)}{\mu(D_j)} \geq 1 - \varepsilon_1',$$

where $\varepsilon_1'(\varepsilon_1) \to 0$ as $\varepsilon_1 \to 0$. By Lemma 2.5,

$$\frac{\mu(\xi_{n_j,j}^{-1}(D_j \cap E_\delta)}{\mu(D_0)} \geq 1 - C\varepsilon_1',$$

for some constant $C > 0$ depending on the ε in Lemma 2.5. We have $C \to 1$ as $\varepsilon \to 0$. Now every parameter $a \in \xi_{n_j,j}^{-1}(D_j \cap E_\delta)$ has that $c_j(a) \in F(f_a)$. For every parameter a in the set

$$A = \bigcap_j \xi_{n_j,j}^{-1}(D_j \cap E_\delta),$$

the critical point $c_j(a) \in F(f_a)$. If $I \neq \{1, \ldots, 2d-2\}$, then there is a small neighbourhood around a in the ball $\mathbb{B}(0,r)$ where all $c_j(a) \in F(f_a)$ for $j \in I$ and, by asssumption (since $x_j \equiv 0$ for $j \neq I$), the other $c_j(a)$ still lands at some hyperbolic set Λ_a. This means that f_a is a J-stable Misiurewicz map. But this contradicts [2]. Hence $I = \{1, \ldots, 2d-2\}$, so every x_j is not identically zero.

Consequently, for every $a \in A$, every $c_j(a) \in F(f_a)$ and it follows that f_a is a hyperbolic map. Since $\varepsilon_1 > 0$ can be chosen arbitrarily small, the Lebesgue density of hyperbolic maps at $a = 0$ is equal to 1 and Theorem B follows.

References

[1] Magnus Aspenberg. Perturbations of rational Misiurewicz maps. Preprint, arXiv:0804.1106.
[2] Magnus Aspenberg. Rational Misiurewicz maps are rare II. Preprint, math.DS/0703306. To appear in Comm. Math. Phys.
[3] Magnus Aspenberg. The Collet-Eckmann condition for rational maps on the Riemann sphere. Ph. D. thesis, Stockholm, 2004. http://www.diva-portal.org/kth/abstract.xsql?dbid=3788. To appear in Math. Z.
[4] Egbert Brieskorn and Horst Knörrer. *Plane algebraic curves*. Birkhäuser Verlag, Basel, 1986.
[5] Lennart Carleson, Peter W. Jones, and Jean-Christophe Yoccoz. Julia and John. *Bol. Soc. Brasil. Mat. (N.S.)*, 25(1):1–30, 1994.
[6] Jacek Graczyk, Grzegorz Światek, and Janina Kotus. Non-recurrent meromorhic functions. *Fund. Math.*, 182:269–281, 2004.
[7] Ricardo Mañé. On a theorem of Fatou. *Bol. Soc. Brasil. Mat. (N.S.)*, 24(1):1–11, 1993.
[8] Nicolae Mihalache. Collet-Eckmann condition for recurrent critical orbits implies uniform hyperbolicity on periodic orbits. *Ergodic Theory Dynam. Systems*, 27(4):1267–1286, 2007.
[9] Michal Misiurewicz. Absolutely continuous invariant measures for certain maps of an interval. *Publ. Math. de l' IHÉS*, 53:17–51, 1981.
[10] Detlef Müller and Fulvio Ricci. Solvability for a class of doubly characteristic differential operators on 2-step nilpotent groups. *Ann. of Math. (2)*, 143(1):1–49, 1996.
[11] Mary Rees. Positive measure sets of ergodic rational maps. *Ann. Sci. École Norm. Sup. (4)*, 19(3):383–407, 1986.
[12] Juan Rivera-Letelier. On the continuity of Hausdorff dimension of Julia sets and similarity between the Mandelbrot set and Julia sets. *Fund. Math.*, 170(3):287–317, 2001.
[13] Mitsuhiro Shishikura and Tan Lei. An alternative proof of Mañé's theorem on non-expanding Julia sets. In *The Mandelbrot set, theme and variations*, volume 274 of *London Math. Soc. Lecture Note Ser.*, pages 265–279. Cambridge Univ. Press, Cambridge, 2000.
[14] Sebastian van Strien. Misiurewicz maps unfold generically (even if they are critically non-finite). *Fund. Math.*, 163(1):39–54, 2000.

Mathematisches Seminar, Christian-Albrechts Univertität zu Kiel, Ludewig-Meyn Str.4, 24 098 Kiel, Germany

E-mail address: aspenberg@math.uni-kiel.de